# A Simple Reduction from a Biased Measure on the Discrete Cube to the Uniform Measure

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**Abstract**

We show that certain statements related to the Fourier-Walsh expansion of functions with respect to a biased measure on the discrete cube can be deduced from the respective results for the uniform measure by a simple reduction. In particular, we present simple generalizations to the biased measure $\mu_p$ of the Bonami-Beckner hypercontractive inequality, and of Talagrand's lower bound on the size of the boundary of subsets of the discrete cube. Our generalizations are tight up to constant factors.

## 1 Introduction

**Definition 1.1.** *Consider the discrete cube $\{0,1\}^n$ endowed with the product measure $\mu_p = (p\delta_{\{1\}} + (1-p)\delta_{\{0\}})^{\otimes n}$, and let $f : \{0,1\}^n \to \mathbb{R}$. The Fourier-Walsh expansion of $f$ with respect to the measure $\mu_p$ is the unique expansion*

$$f = \sum_{S \subset \{1,2,\ldots,n\}} \alpha_S u_S,$$

*where for any $T \subset \{1,2,\ldots,n\}$,*[1]

$$u_S(T) = \Big( -\sqrt{\frac{1-p}{p}} \Big)^{|S \cap T|} \Big( \sqrt{\frac{p}{1-p}} \Big)^{|S \setminus T|}.$$

*In particular, for the uniform measure (i.e., $p = 1/2$), $u_S(T) = (-1)^{|S \cap T|}$. The coefficients $\alpha_S$ are denoted by $\hat{f}(S)$.*[2]

Properties of the Fourier-Walsh expansion are one of the main objects of study in discrete harmonic analysis. Many of the results in this field were obtained for the uniform measure

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[1]Throughout the paper, we identify elements of $\{0,1\}^n$ with subsets of $\{1,2,\ldots,n\}$ in the natural way.

[2]Note that since the functions $\{u_S\}_{S \subset \{1,\ldots,n\}}$ form an orthonormal basis, the representation is indeed unique, and the coefficients are given by the formula $\hat{f}(S) = \mathbb{E}_{\mu_p}[f \cdot u_S]$ .

on the discrete cube. However, various applications (including applications to random graph theory [12], to hardness of approximation [8], and to other areas) require consideration of a biased product measure on the discrete cube. This led to a series of papers generalizing results from the uniform measure $\mu_{1/2}$ to general biased measures $\mu_p$ (see, e.g., [10, 14, 21, 22]). The usual way to obtain such generalizations is to repeat the original proof, replacing the analytic tools used in the uniform case (like the Bonami-Beckner hypercontractive inequality [4, 1]) by their biased analogues. This approach is effective, and is already considered quite standard. However, it still requires thorough examination and adaptation of the (sometimes complex) proofs of the results in the uniform measure case.

In this paper we study a simple reduction from the biased measure $\mu_p$ to the uniform measure $\mu_{1/2}$. We note that this reduction was already considered in several papers (e.g., [5]). We discuss the previous results and compare them with our work in Section 4.

Assume that $p = t/2^m$.[3] For any function $f : \{0,1\}^n \to \mathbb{R}$ we define a function $Red(f) = g : \{0,1\}^{mn} \to \mathbb{R}$ as follows: each $y \in \{0,1\}^{mn}$ is considered as a concatenation of $n$ vectors $y^i \in \{0,1\}^m$, and each such vector is translated to a natural number $0 \leq Bin(y^i) < 2^m$ through its binary expansion (i.e., $Bin(y^i) = \sum_{j=0}^{m-1} 2^j \cdot y^i_{m-j}$). Then, for any $y \in \{0,1\}^{mn}$,

$$g(y) = g(y^1, y^2, \ldots, y^n) := f\left(h(y^1), h(y^2), \ldots, h(y^n)\right),$$

where $h : \{0,1\}^m \to \{0,1\}$ is given by

$$h(y^i) = \begin{cases} 1, & Bin(y^i) \geq 2^m - t \\ 0, & Bin(y^i) < 2^m - t. \end{cases}$$

It is clear from the construction that the expectation of $g$ w.r.t. the uniform measure is equal to the expectation of $f$ w.r.t. the measure $\mu_p$. Furthermore, it was shown in [12] that the sum of influences[4] of $g$ can be bounded *from above* by a simple function of the sum of influences of $f$. This allows to generalize to the biased measure statements concerning lower bounds on the sum of influences, such as the KKL theorem [17].

We show that for any $d$, the Fourier weight of $g$ on the $d$-th level (i.e., $\sum_{|S|=d} \hat{g}(S)^2$) is bounded *from below* in terms of the Fourier weight of $f$ on the $d$-th level:

**Theorem 1.2.** *Consider the discrete cube $\{0,1\}^n$ endowed with the product measure $\mu_p$ for $p = t/2^m \leq 1/2$. Let $f : \{0,1\}^n \to \mathbb{R}$, and let $Red(f) = g : \{0,1\}^{mn} \to \mathbb{R}$ be obtained from $f$ by the construction described above. For any $1 \leq d \leq n$,*

$$\sum_{|S|=d} \hat{g}(S)^2 \geq \left(\frac{p\lfloor \log(1/p) \rfloor}{1-p}\right)^d \sum_{|S|=d} \hat{f}(S)^2,$$

*where the Fourier-Walsh coefficients of $g$ are w.r.t. the uniform measure and the coefficients of $f$ are w.r.t. the measure $\mu_p$.*

Combination of Theorem 1.2 with the results of [12] allows to generalize to the biased measure statements concerning upper bounds on the Fourier weight of the functions at any level, and statements combining such upper bounds with lower bounds on the sum of influences. For example, we obtain the following generalization of the Bonami-Beckner hypercontractive inequality – probably the most widely used tool in discrete harmonic analysis.

[3] It is clear that there is no loss of generality in assuming that $p$ is diadic, as the results for general $p$ follow immediately by approximation. The exact statement of our main result for a general $p$ is given in Proposition 2.3.

[4] See Section 2.2 for the definition of influences.

**Proposition 1.3.** *Consider the discrete cube $\{0,1\}^n$ endowed with the product measure $\mu_p$, $p \leq 1/2$, and let $T_\delta$ denote the noise operator with rate $\delta$, defined by $T_\delta f = \sum_S \delta^{|S|} \hat{f}(S) u_S$. For any function $f : \{0,1\}^n \to \mathbb{R}$ and for any $0 \leq \delta \leq \sqrt{\frac{p\lfloor \log(1/p) \rfloor}{1-p}}$,*

$$||T_\delta f||_2 \leq ||f||_{1+\frac{1-p}{p\lfloor \log(1/p)\rfloor}\delta^2}.$$

Another example is a lower bound on the size of the vertex boundary of subsets of the discrete cube obtained by Talagrand [28].

**Proposition 1.4.** *Consider the discrete cube $\{0,1\}^n$ endowed with the product measure $\mu_p$, $p \leq 1/2$. There exists $\alpha > 0$ such that for any monotone subset $A \subset \{0,1\}^n$,*

$$\mu_p(\partial A) \sum_{i=1}^{n} \mu_p(A_i) \geq \frac{c}{\lfloor \log(1/p) \rfloor} \varphi\left(\mu_p(A)(1-\mu_p(A))\right) \psi\left(3\lfloor \log(1/p)\rfloor \sum_{i=1}^{n} \mu_p(A_i)^2\right), \qquad (1)$$

*where $\partial A$ is the vertex boundary of $A$, $A_i$ is the vertex boundary of $A$ in the $i$'th direction, $\varphi(x) = x^2[\log(e/x)]^{1-\alpha}$, $\psi(x) = [\log(e/x)]^{\alpha}$, and $c > 0$ is a universal constant.*

Both Proposition 1.3 and Proposition 1.4 are tight, up to constant factors.

We note that the main difference between the reduction technique and the standard proof strategy presented above is that the reduction allows to transform *the statement* directly to the biased case, without considering the proof. In cases where the proof in the uniform case is complex, like in the two propositions above, the reduction simplifies the generalization to the biased case significantly.

This paper is organized as follows: The proofs of Theorem 1.2 and of several other properties of the simple reduction are presented in Section 2. In Section 3 we prove Propositions 1.3 and 1.4, and present several other applications. Finally, we compare our results with previous work and raise questions for further research in Section 4.

# 2 The Reduction

## 2.1 Lower Bound on the Fourier Weight of $g = Red(f)$ on Fixed Levels

In this subsection we present the proof of Theorem 1.2. For the sake of clarity, we consider first the case $p = 1/2^m$, where we establish a simple relation between the Fourier-Walsh coefficients of $g$ and the respective coefficients of $f$. Then we generalize the proof to any diadic $p$, and finally we give a slightly weaker formulation of the theorem that holds for a general $p$.

Throughout the proof, $y = (y^1, y^2, \ldots, y^n)$ denotes an element of $\{0,1\}^{mn}$. (Note that superscript indices, like $y^i$, denote vectors in $\{0,1\}^m$, rather than single coordinates.) The functions we consider are $f : \{0,1\}^n \to \mathbb{R}$ and $Red(f) = g : \{0,1\}^{mn} \to \mathbb{R}$. All the computations related to $f$ are w.r.t. the measure $\mu_p$, and all the computations related to $g$ are w.r.t. the uniform measure $\mu_{1/2}$.

### 2.1.1 The case $p = 1/2^m$

In this case, the Fourier-Walsh coefficients of $g$ can be expressed as a simple function of the coefficients of $f$.

**Proposition 2.1.** *Assume that $p = 1/2^m$. For any $S \subset \{1, 2, \dots, mn\}$, denote $S_i = S \cap \{(i-1)m+1, (i-1)m+2, \dots, im\}$. Let $S' = \{i : |S_i| > 0\}$, and denote $k = |S'|$. Then:*

$$\hat{g}(S) = \left(-\sqrt{\frac{p}{1-p}}\right)^k (-1)^{|S|} \hat{f}(S').$$

**Proof:** First, we note that since both the Fourier transform and the construction of $g$ are linear, we have (for all $f_1, f_2, \alpha$, and $\beta$), that if $g = Red(\alpha \cdot f_1 + \beta \cdot f_2)$, then for any $S$,

$$\hat{g}(S) = \alpha \cdot \widehat{Red(f_1)}(S) + \beta \cdot \widehat{Red(f_2)}(S).$$

Thus, it is sufficient to prove the assertion for the characters $\{u_T\}_{T \subset \{1,\dots,n\}}$, which form a basis for the space of all functions from $\{0,1\}^n$ to $\mathbb{R}$.

Second, we note that by the structure of the characters, if $f = u_T$ and $g = Red(f)$, then:

$$g(y^1, y^2, \dots, y^n) = u_T\left(h(y^1), h(y^2), \dots, h(y^n)\right) = \prod_{i \in T} u_{\{i\}}(h(y^i)).$$

Hence, we can "decompose" the function $g$ into functions $\{g_i : \{0,1\}^m \to \mathbb{R}\}_{i=1,2,\dots,n}$, defined by $g_i(y^i) = u_{\{i\}}(h(y^i))$, and then by the properties of the Fourier transform, we have that for any $S$,

$$\hat{g}(S) = \prod_{i \in T} \hat{g}_i(S_i).$$

Therefore, it is sufficient to prove the assertion for the characters $u_{\{i\}}$ for $i = 1, 2, \dots, n$, and the result follows by multiplicativity.

Third, by symmetry, it is sufficient to consider $i = 1$, and thus we may assume w.l.o.g. that $S \subset \{1, 2, \dots, m\}$. In this case, since we clearly have $\widehat{u_{\{1\}}}(\{1\}) = 1$, the assertion of the proposition is reduced to the following:

$$\hat{g}_1(S) = \left(-\sqrt{\frac{p}{1-p}}\right) \cdot (-1)^{|S|}, \tag{2}$$

where $g_1 = Red(u_{\{1\}})$.

Finally, Equation (2) is obtained by simple computation. Indeed, since $p = 1/2^m$, by the definition of the reduction we have $g_1(y^1) = -\sqrt{(1-p)/p}$ if $y^1 = (1, 1, \dots, 1)$ and $g_1(y^1) = \sqrt{p/(1-p)}$ otherwise. Since for any $S \subset \{1, 2, \dots, n\}$ we have $\mathbb{E}[u_S] = 0$, we can write:

$$\begin{aligned}
\hat{g}_1(S) &= \mathbb{E}_{y^1 \in \{0,1\}^m}(g_1(y^1) \cdot u_S(y^1)) = \mathbb{E}_{y^1 \in \{0,1\}^m}((g_1(y^1) - \sqrt{p/(1-p)}) \cdot u_S(y^1)) \\
&= \frac{1}{2^m} \cdot (-\sqrt{(1-p)/p} - \sqrt{p/(1-p)}) \cdot u_S((1,1,\dots,1)) = p \cdot \frac{-1}{\sqrt{p(1-p)}} \cdot (-1)^{|S|} \\
&= \left(-\sqrt{\frac{p}{1-p}}\right) \cdot (-1)^{|S|},
\end{aligned} \tag{3}$$

as asserted. $\square$

### 2.1.2 The case $p = t/2^m$

In this case, the relation between the Fourier-Walsh coefficients of $g$ and the corresponding coefficients of $f$ is a bit more complex.

**Proposition 2.2.** *Assume that $p = t/2^m$. For $j = 1, 2, \ldots, m$, let*

$$a_j(t) = \min\left(t \ mod\ 2^{m-j+1}, 2^{m-j+1} - t \ mod\ 2^{m-j+1}\right).$$

*For any $S \subset \{1, 2, \ldots, mn\}$, denote $S_i = S \cap \{(i-1)m+1, (i-1)m+2, \ldots, im\}$, and $s_i = \max(S_i) - (i-1)m$. Let $S' = \{i : |S_i| > 0\}$, and denote $k = |S'|$. We have*

$$|\hat{g}(S)| = \left(\prod_{i \in S'} \frac{a_{s_i}(t)}{t}\right) \cdot \left(\sqrt{\frac{p}{1-p}}\right)^k |\hat{f}(S')|. \tag{4}$$

*Furthermore, if for all $1 \leq i \leq n$, $s_i \leq \lfloor \log(1/p) \rfloor$, then:*

$$\hat{g}(S) = \left(-\sqrt{\frac{p}{1-p}}\right)^k (-1)^{|S|} \hat{f}(S'). \tag{5}$$

**Proof:** As in the case $p = 1/2^m$, it is sufficient to prove the assertion for $f = u_{\{1\}}$ and $S \subset \{1, 2, \ldots, m\}$. Denote $Red(u_{\{1\}}) = g_1$. By the definition of the reduction, we have $g_1(y^1) = -\sqrt{(1-p)/p}$ if $Bin(y^1) \geq 2^m - t$ and $g_1(y^1) = \sqrt{p/(1-p)}$ otherwise. Thus, a computation similar to that given in Equation (3) shows that for any $S \subset \{1, 2, \ldots, m\}$,

$$\begin{aligned}\hat{g}_1(S) &= \sum_{\{y^1 : Bin(y^1) \geq 2^m - t\}} 2^{-m} \cdot \frac{-1}{\sqrt{p(1-p)}} \cdot u_S(y^1) \\ &= \frac{1}{t} \cdot \left(-\sqrt{\frac{p}{1-p}}\right) \cdot \sum_{\{y^1 : Bin(y^1) \geq 2^m - t\}} u_S(y^1).\end{aligned}$$

Thus, Equation (4) would follow once we show that

$$\left| \sum_{\{y^1 : Bin(y^1) \geq 2^m - t\}} u_S(y^1) \right| = a_s(t), \tag{6}$$

where $s = \max(S)$.

In order to compute the left hand side of Equation (6), we note that for any $l \in \mathbb{N}$,

$$\sum_{\{y^1 : l \cdot 2^{m-s+1} \leq Bin(y^1) < (l+1)2^{m-s+1}\}} u_S(y^1) = 0.$$

Indeed, each such sequence of $2^{m-s+1}$ consecutive values of $y^1$ is composed of the two sequences

$$\{y^1 : 2l \cdot 2^{m-s} \leq Bin(y^1) < (2l+1)2^{m-s}\} \quad \text{and} \quad \{y^1 : (2l+1)2^{m-s} \leq Bin(y^1) < (2l+2)2^{m-s}\}.$$

Inside each of the sequences, the vectors $y^1$ differ only in coordinates that are not included in $S$, and thus the value of $u_S(y^1)$ is equal for all elements of the sequence. The only difference

between the sequences is in the $s$'s coordinate that is included in $S$, and hence, the sums of $u_S(y^1)$ over the sequences cancel each other. Thus, due to the cancelation we have:

$$\sum_{\{y^1:Bin(y^1)\geq 2^m-t\}} u_S(y^1) = \sum_{y^1\in V} u_S(y^1),$$

where

$$V = \{y^1 : 2^m - t \leq Bin(y^1) \leq 2^m - t + \big((t-1) \bmod 2^{m-s+1}\big)\}.$$

Using the same argument we see that if ($t \bmod 2^{m-j+1} \geq 2^{m-j}$), then the value of $u_S(y^1)$ is the same for all $y^1 \in V$, and thus,

$$\left|\sum_{y^1\in V} u_S(y^1)\right| = 2^{m-s+1} - (t \bmod 2^{m-s+1}).$$

Similarly, if ($t \bmod 2^{m-s+1} < 2^{m-s}$), then part of the elements of the sum cancel each other, and we obtain:

$$\left|\sum_{y^1\in V} u_S(y^1)\right| = t \bmod 2^{m-s+1}.$$

Combining the two cases, we get:

$$\left|\sum_{\{y^1:Bin(y^1)\geq 2^m-t\}} u_S(y^1)\right| = \left|\sum_{y^1\in V} u_S(y^1)\right| = \min\big(t \bmod 2^{m-s+1}, 2^{m-s+1} - t \bmod 2^{m-s+1}\big) = a_s(t), \tag{7}$$

proving Equation (6).

If $\max(S) \leq \lfloor\log(1/p)\rfloor$, the expression is much simpler. We note that for all $1 \leq j \leq \lfloor\log(1/p)\rfloor$, the $j$-th coordinate of all the vectors $y^1 \in \{0,1\}^m$ such that $Bin(y^1) \geq 2^m - t$, is one. Thus, if $S \subset \{1, 2, \ldots, \lfloor\log(1/p)\rfloor\}$, then $u_S(y^1) = (-1)^{|S|}$ for all $y^1$ with $Bin(y^1) \geq 2^m - t$. Therefore, in this case we have:

$$\sum_{\{y^1:Bin(y^1)\geq 2^m-t\}} u_S(y^1) = (-1)^{|S|} \cdot t, \tag{8}$$

and this implies Equation (5), completing the proof of Proposition 2.2. □

Theorem 1.2 follows immediately from Proposition 2.2. Indeed, setting:

$$A_{\{i_1,i_2,\ldots,i_k\}} = \{S \subset \{1,\ldots,n\} : (|S| = k) \wedge (S' = \{i_1,\ldots,i_k\}) \wedge (\forall i \in S', s_i \leq \lfloor\log(1/p)\rfloor)\},$$

we have (by Equation (5)), that:

$$\begin{aligned}
\sum_{|S|=k} \hat{g}(S)^2 \geq & \sum_{\{i_1,i_2,\ldots,i_k\}\subset\{1,\ldots,n\}} \left(\sum_{S\in A_{\{i_1,i_2,\ldots,i_k\}}} \hat{g}(S)^2\right) \\
= & \sum_{\{i_1,i_2,\ldots,i_k\}\subset\{1,\ldots,n\}} (\lfloor\log(1/p)\rfloor)^k \left(\frac{p}{1-p}\right)^k \hat{f}(\{i_1,i_2,\ldots,i_k\})^2 \\
= & \left(\frac{p\lfloor\log(1/p)\rfloor}{1-p}\right)^k \sum_{|S'|=k} \hat{f}(S')^2,
\end{aligned}$$

as asserted.

### 2.1.3 The Non-diadic Case

In the case of a non-diadic $p$, we can use approximation to get a slightly weaker variant of Theorem 1.2.

**Proposition 2.3.** *Consider the discrete cube $\{0,1\}^n$ endowed with the product measure $\mu_p$, $p \leq 1/2$. For any function $f : \{0,1\}^n \to \mathbb{R}$, and for any $\epsilon > 0$, there exists $m \in \mathbb{N}$ and a function $g : \{0,1\}^{mn} \to \mathbb{R}$, such that:*

- $|\mathbb{E}[g] - \mathbb{E}[f]| < \epsilon$, *and*
- *For all* $1 \leq d \leq n$,

$$\sum_{|S|=d} \hat{g}(S)^2 \geq \Big(\frac{p\lfloor \log(1/p) \rfloor}{1-p}\Big)^d \sum_{|S|=d} \hat{f}(S)^2 - \epsilon,$$

*where the expectation and the Fourier-Walsh coefficients of $g$ are w.r.t. the uniform measure and the expectation and the coefficients of $f$ are w.r.t. the measure $\mu_p$.*

**Proof:** For any function $f : \{0,1\}^n \to \mathbb{R}$, the maps $p \mapsto \mathbb{E}_{\mu_p}(f)$ and $p \mapsto \sum_{|S|=d} \hat{f}_{\mu_p}(S)^2$ (where $\hat{f}_{\mu_p}(S)$ denotes coefficients w.r.t. the measure $\mu_p$) are uniformly continuous (as functions of $p$) in $[0,1]$. Therefore, for any $\epsilon > 0$, we can choose a diadic $p' = t/2^m$ close enough to $p$ such that the function $g : \{0,1\}^{mn} \to \mathbb{R}$ constructed from $f$ by the reduction procedure w.r.t. the measure $\mu_{p'}$ will satisfy the conditions of the proposition. $\square$

## 2.2 Upper Bound on the Influences of $g$

Before we turn to the applications, we present the result of Friedgut and Kalai [12] mentioned in the introduction that allows to bound the sum of influences of $g$ from above in terms of the influences of $f$, and obtain a variant of that result that allows to bound the sum of squares of the influences. We note that while the result of [12] generalizes to the continuous cube with the Lebesgue measure (as shown in [19]), the proof of the bound on the squares of influences holds only in the discrete setting.

**Definition 2.4.** *Consider the discrete cube $\{0,1\}^n$ endowed with the product measure $\mu_p$. Let $f : \{0,1\}^n \to \{0,1\}$. For $1 \leq i \leq n$, the influence of the $i$-th coordinate on $f$ is*

$$I_i(f) = \Pr_{x \sim \mu_p}[f(x) \neq f(x \oplus e_i)],$$

*where $x \oplus e_i$ denotes the vector obtained from $x$ by replacing $x_i$ by $1 - x_i$ and leaving the other coordinates unchanged.*

*For a set $A \subset \{0,1\}^n$, we define $I_i(A) = I_i(1_A)$.*

**Theorem 2.5** (Friedgut and Kalai)**.** *Consider the discrete cube $\{0,1\}^n$ endowed with the product measure $\mu_p$, for $p = t/2^m \leq 1/2$. Let $f : \{0,1\}^n \to \{0,1\}$, and let $g : \{0,1\}^{mn} \to \{0,1\}$ be obtained from $f$ by the construction described in the introduction. Then*

$$\sum_{i=1}^{mn} I_i(g) \leq 6p\lfloor \log(1/p) \rfloor \sum_{i=1}^{n} I_i(f),$$

*where the influences on $f$ are w.r.t. $\mu_p$, and the influences on $g$ are w.r.t. $\mu_{1/2}$.*

**Proposition 2.6.** *Let $f, g$ be as defined in Theorem 2.5. Then*

$$\sum_{i=1}^{mn} I_i(g)^2 \leq 12p^2\lfloor\log(1/p)\rfloor \sum_{i=1}^{n} I_i(f)^2.$$

The proof is essentially the same as the proof of Theorem 2.5 given in [12]. For the sake of completeness we present it here.

**Proof:** For $1 \leq i \leq n$, consider the influences of the coordinates $(i-1)m+1, \ldots, im$ on $g$. It is easy to see that[5]

$$I_{(i-1)m+j}(g) \leq \begin{cases} 2p \cdot I_i(f), & j \leq \lfloor\log(1/p)\rfloor, \\ 2^{-j+2} \cdot I_i(f), & j > \lfloor\log(1/p)\rfloor. \end{cases}$$

Thus,

$$\begin{aligned}\sum_{j=1}^{m} I_{(i-1)m+j}(g)^2 &\leq \left(\lfloor\log(1/p)\rfloor \cdot 4p^2 + \sum_{j=\lfloor\log(1/p)\rfloor+1}^{m} 2^{-2j+4}\right) I_i(f)^2 \\ &\leq (4p^2\lfloor\log(1/p)\rfloor + 8p^2) I_i(f)^2 \leq 12p^2\lfloor\log(1/p)\rfloor I_i(f)^2.\end{aligned}$$

Summing over $i$ completes the proof. $\square$

Since for a fixed function $f : \{0,1\}^n \to \{0,1\}$, the maps $p \mapsto \sum_i I_i^p(f)$ and $p \mapsto \sum_i I_i^p(f)^2$ (where $I_i^p$ denotes influence w.r.t. $\mu_p$) are uniformly continuous as function of $p$ in $[0,1]$, we immediately get the following:

**Proposition 2.7.** *Consider the discrete cube $\{0,1\}^n$ endowed with the product measure $\mu_p$, $p \leq 1/2$. For any function $f : \{0,1\}^n \to \{0,1\}$, and for any $\epsilon > 0$, there exists $m \in \mathbb{N}$ and a function $g : \{0,1\}^{mn} \to \mathbb{R}$, such that:*

- $|\mathbb{E}[g] - \mathbb{E}[f]| < \epsilon$,
- $\sum_{i=1}^{mn} I_i(g) \leq 6p\lfloor\log(1/p)\rfloor \sum_{i=1}^{n} I_i(f) + \epsilon$, *and*
- $\sum_{i=1}^{mn} I_i(g)^2 \leq 12p^2\lfloor\log(1/p)\rfloor \sum_{i=1}^{n} I_i(f)^2 + \epsilon$,

*where the expectation and the influences of $g$ are w.r.t. the uniform measure and the expectation and the influences of $f$ are w.r.t. the measure $\mu_p$.*

Furthermore, it is clear that $g$ can be chosen such that it will satisfy the assertions of Propositions 2.3 and 2.7 simultaneously.

**Remark 2.8.** *We note that using Proposition 2.2, the proof of Proposition 2.6 can be easily adapted to show that*

$$\sum_{i=1}^{mn} \hat{g}(\{i\})^2 \leq 3\frac{p\lfloor\log(1/p)\rfloor}{1-p} \sum_{i=1}^{n} \hat{f}(\{i\})^2, \tag{9}$$

*proving the tightness of Theorem 1.2 for $d = 1$ up to a multiplicative constant.*

[5] We note that this part of the argument does not hold for the Lebesgue measure on the continuous cube. It does hold for any single fiber (w.r.t. the expectation on that fiber), but unlike the case of influences where the fibers can be combined using Fubini's theorem, in the case of squares of influences the fibers cannot be combined.

# 3 Applications

In this section we apply Theorem 1.2 to obtain simple generalizations to the biased measure $\mu_p$ of several results:

- The Bonami-Beckner hypercontractive inequality [4, 1],
- A relation between the size of the vertex boundary of a monotone[6] subset of the discrete cube and its influences, obtained by Talagrand [28],
- An upper bound on the $d$-th level Fourier-Walsh coefficients of a monotone function in terms of its influences, obtained by Talagrand [27] for $d = 2$ and generalized by Benjamini et al. [2] to any $d$, and
- A lower bound on the correlation between monotone families, obtained by Talagrand [27].

Throughout this section, we assume for the sake of simplicity that $p$ is diadic (i.e., $p = t/2^m$), and satisfies $p \leq 1/2$. All the proofs generalize immediately to a non-diadic $p$ by choosing a diadic $p'$ "close enough" to $p$, replacing Theorems 1.2 and 2.5 by Propositions 2.3 and 2.7 (respectively), and considering the limit as $\epsilon \to 0$. The case $p > 1/2$ also follows immediately by considering a variant of the *dual function* defined as

$$f'(x_1, x_2, \ldots, x_n) = f(1 - x_1, 1 - x_2, \ldots, 1 - x_n),$$

and noting that for any $S \subset \{1, \ldots, n\}$, we have $\hat{f}'(S) = (-1)^{|S|} \hat{f}(S)$, where the Fourier-Walsh coefficients of $f$ are w.r.t. $\mu_p$ and the coefficients of $f'$ are w.r.t. $\mu_{1-p}$.

For a function $f : \{0,1\}^n \to \mathbb{R}$, the function $Red(f) = g : \{0,1\}^{mn} \to \mathbb{R}$ denotes the function obtained from $f$ by the reduction procedure described in the introduction. All the computations related to $f$ are w.r.t. the measure $\mu_p$, and all the computations related to $g$ are w.r.t. the uniform measure.

## 3.1 The Bonami-Beckner Hypercontractive Inequality

One of the main tools in discrete harmonic analysis is the Bonami-Beckner hypercontractive inequality [4, 1]. The inequality considers a special operator called the *noise operator* that has a simple description in terms of the Fourier-Walsh expansion.

**Definition 3.1.** *For a function $f : \{0,1\}^n \to \mathbb{R}$ with Fourier-Walsh expansion $f = \sum_S \hat{f}(S) u_S$, the application of the noise operator with rate $\delta$ to $f$ is*

$$T_\delta f = \sum_S \delta^{|S|} \hat{f}(S) u_S.$$

**Theorem 3.2** (Bonami,Beckner)**.** *Consider the discrete cube $\{0,1\}^n$ endowed with the uniform measure. For any function $f : \{0,1\}^n \to \mathbb{R}$ and for any $0 \leq \delta \leq 1$,*

$$||T_\delta f||_2 \leq ||f||_{1+\delta^2}.$$

Using Theorem 1.2, we get the following generalization to a biased measure $\mu_p$:

[6] A function $f : \{0,1\}^n \to \mathbb{R}$ is monotone if for all $x, y \in [0,1]^n$, $\forall i (x_i \geq y_i) \implies f(x) \geq f(y)$. A subset $A \subset \{0,1\}^n$ is monotone if its characteristic function $1_A$ is monotone.

**Proposition 3.3.** *Consider the discrete cube $\{0,1\}^n$ endowed with the product measure $\mu_p$, $p \leq 1/2$. For any function $f : \{0,1\}^n \to \mathbb{R}$ and for any $0 \leq \delta \leq \sqrt{\frac{p\lfloor \log(1/p) \rfloor}{1-p}}$,*

$$||T_\delta f||_2 \leq ||f||_{1+\frac{1-p}{p\lfloor \log(1/p) \rfloor}\delta^2}.$$

**Proof:** We have

$$||T_\delta f||_2^2 = \sum_S \delta^{2|S|} \hat{f}(S)^2 = \sum_{d=0}^{n} \delta^{2d} \sum_{|S|=d} \hat{f}(S)^2 \leq \sum_{d=0}^{n} \delta^{2d} \cdot \left( \frac{1-p}{p\lfloor \log(1/p) \rfloor} \right)^d \sum_{|S|=d} \hat{g}(S)^2$$
$$\leq \sum_{d=0}^{mn} \left( \delta \sqrt{\frac{1-p}{p\lfloor \log(1/p) \rfloor}} \right)^{2d} \sum_{|S|=d} \hat{g}(S)^2 = ||T_{\delta\sqrt{\frac{1-p}{p\lfloor \log(1/p) \rfloor}}} g||_2^2$$
$$\leq ||g||^2_{1+\frac{1-p}{p\lfloor \log(1/p) \rfloor}\delta^2} = ||f||^2_{1+\frac{1-p}{p\lfloor \log(1/p) \rfloor}\delta^2}.$$

The first and the third equalities follow from the Parseval identity, the first inequality follows from Theorem 1.2, the third inequality follows from the Bonami-Beckner inequality (for the uniform measure), and the last equality follows since by the construction of $g$, we have $||f||_q = ||g||_q$ for any norm $q$. □

By the duality of $L^p$ norms, Proposition 3.3 implies:

**Proposition 3.4.** *Consider the discrete cube $\{0,1\}^n$ endowed with the product measure $\mu_p$, $p \leq 1/2$. If a function $f : \{0,1\}^n \to \mathbb{R}$ satisfies $\hat{f}(S) = 0$ for all $|S| > d$, then for any $q \geq 2$,*

$$||f||_q \leq \left( \frac{1-p}{p\lfloor \log(1/p) \rfloor} \cdot (q-1) \right)^{d/2} ||f||_2.$$

We note that the problem of finding the optimal hypercontractivity constant for a biased measure $\mu_p$, i.e., finding the minimal value $C_{p,q}$ for which in the assumptions of Proposition 3.4,

$$||f||_q \leq (C_{p,q})^{d/2} ||f||_2,$$

was studied in various works, in several different contexts. Partial results were obtained by Talagrand [26], Friedgut [10] and Kindler [22] and applied in the study of Boolean functions. The optimal value of $C_{p,q}$ is attributed to Rothaus (unpublished), was stated without proof in lecture notes of Higuchi and Yoshida [16], and given with proof by Diaconis and Saloff-Coste [7] in the context of the logarithmic Sobolev inequality. The formulation we use in Proposition 3.4 was used by Oleszkiewicz [24] in the context of the Khinchine-Kahane inequality. For $q \geq \ln(1/p)$ (which is usually the case in applications), the value of $C_{p,q}$ obtained in Proposition 3.4 matches the optimal value given in Theorem 2.1 of [24], up to a multiplicative constant.

## 3.2 Relation Between the Influences and the Size of the Boundary

**Definition 3.5.** *Consider the discrete cube $\{0,1\}^n$ endowed with the product measure $\mu_p$. For a monotone set $A \subset \{0,1\}^n$ and $1 \leq i \leq n$, let*

$$A_i = \{x \in A : x \oplus e_i \notin A\}.^7$$

[7] Note that this definition is closely related to the notion of influences, as for any monotone set $A$, we have $\mu_p(A_i) = p \cdot I_i(A)$, where the influence is w.r.t. the measure $\mu_p$.

*The vertex boundary of $A$ is*

$$\partial A = \bigcup_{i=1}^{n} A_i = \{x \in A : \exists (1 \le i \le n), x \oplus e_i \notin A\}.$$

In [23], Margulis proved that for subsets of the discrete cube endowed with the uniform measure, the size of the boundary and the sum of influences cannot be small simultaneously. For monotone subsets of the discrete cube, Talagrand [28] gave the following precise form to this statement:

**Theorem 3.6** (Talagrand)**.** *Consider the discrete cube $\{0,1\}^n$ endowed with the uniform measure $\mu_{1/2}$. There exists $\alpha > 0$ such that for any monotone subset $A \subset \{0,1\}^n$,*

$$\mu_{1/2}(\partial A) \sum_{i=1}^{n} \mu_{1/2}(A_i) \ge c\varphi\left(\mu_{1/2}(A)(1-\mu_{1/2}(A))\right) \psi\left(\sum_{i=1}^{n} \mu_{1/2}(A_i)^2\right),$$

*where $\varphi(x) = x^2[\log(e/x)]^{1-\alpha}$, $\psi(x) = [\log(e/x)]^{\alpha}$, and $c > 0$ is a universal constant.*

Using Proposition 2.6, we get the following generalization to the measure $\mu_p$:

**Proposition 3.7.** *Consider the discrete cube $\{0,1\}^n$ endowed with the product measure $\mu_p$, $p \le 1/2$. There exists $\alpha > 0$ such that for any monotone subset $A \subset \{0,1\}^n$,*

$$\mu_p(\partial A) \sum_{i=1}^{n} \mu_p(A_i) \ge \frac{c}{\lfloor \log(1/p) \rfloor} \varphi\left(\mu_p(A)(1-\mu_p(A))\right) \psi\left(3\lfloor \log(1/p) \rfloor \sum_{i=1}^{n} \mu_p(A_i)^2\right), \tag{10}$$

*where $\varphi(x) = x^2[\log(e/x)]^{1-\alpha}$, $\psi(x) = [\log(e/x)]^{\alpha}$, and $c > 0$ is a universal constant.*

**Proof:** Denote $f = 1_A$, construct $g : \{0,1\}^{mn} \to \{0,1\}$ as described in the introduction, and let $B = \{y \in \{0,1\}^{mn} : g(y) = 1\}$. It is easy to see that $\mu_{1/2}(\partial B) \le \mu_p(\partial A)$. Indeed, to any $y = (y^1, y^2, \ldots, y^n) \in \{0,1\}^{mn}$ we can attach $H(y) \in \{0,1\}^n$ by setting $H(y)_i = 1$ if $y^i \neq (0,0,\ldots,0)$ and $H(y)_i = 0$ otherwise, and then it is clear by the construction of $g$ that $(y \in \partial(B)) \Rightarrow (H(y) \in \partial(A))$. Since for any $x \in A$ we have $\mu_{1/2}(\{y \in B : H(y) = x\}) = \mu_p(x)$, it follows that $\mu_{1/2}(\partial B) \le \mu_p(\partial A)$.

Furthermore, it follows from Theorem 2.5 that

$$\sum_{j=1}^{mn} \mu_{1/2}(B_j) \le 3\lfloor \log(1/p) \rfloor \sum_{i=1}^{n} \mu_p(A_i).$$

Therefore,

$$\mu_{1/2}(\partial B) \sum_{j=1}^{mn} \mu_{1/2}(B_j) \le 3\lfloor \log(1/p) \rfloor \mu_p(\partial A) \sum_{i=1}^{n} \mu_p(A_i). \tag{11}$$

On the other hand, it follows from Proposition 2.6 that

$$\sum_{j=1}^{mn} \mu_{1/2}(B_j)^2 \le 3\lfloor \log(1/p) \rfloor \sum_{i=1}^{n} \mu_p(A_i)^2.$$

Since by the construction of $B$, we have $\mu_{1/2}(B) = \mu_p(A)$, and since the function $\psi(x) = \log(e/x)^\alpha$ is monotone decreasing, it follows that

$$\varphi\left(\mu_{1/2}(B)(1-\mu_{1/2}(B))\right)\psi\left(\sum_{j=1}^{mn}\mu_{1/2}(B_j)^2\right) \geq \varphi\left(\mu_p(A)(1-\mu_p(A))\right)\psi\left(3\lfloor\log(1/p)\rfloor\sum_{i=1}^{n}\mu_p(A_i)^2\right). \tag{12}$$

Combining Equations (11) and (12) with Theorem 3.6, we get

$$\begin{aligned}
\mu_p(\partial A)\sum_{i=1}^{n}\mu_p(A_i) &\geq \frac{1}{3\lfloor\log(1/p)\rfloor}\mu_{1/2}(\partial B)\sum_{j=1}^{mn}\mu_{1/2}(B_j) \\
&\geq \frac{c}{3\lfloor\log(1/p)\rfloor}\varphi\left(\mu_{1/2}(B)(1-\mu_{1/2}(B))\right)\psi\left(\sum_{j=1}^{mn}\mu_{1/2}(B_j)^2\right) \\
&\geq \frac{c'}{\lfloor\log(1/p)\rfloor}\varphi\left(\mu_p(A)(1-\mu_p(A))\right)\psi\left(3\lfloor\log(1/p)\rfloor\sum_{i=1}^{n}\mu_p(A_i)^2\right),
\end{aligned}$$

as asserted. □

We show now that Proposition 3.7 is tight by considering a balanced threshold function on the biased discrete cube. Consider the measure $\mu_p$ on $\{0,1\}^n$, and let

$$A = \{x \in \{0,1\}^n : \sum_{i=1}^{n} x_i > \lfloor np\rfloor\}.$$

It is well-known that $\mu_p(A) = \Theta(1)$. We have

$$\partial A = \{x \in \{0,1\}^n : \sum_{i=1}^{n} x_i = \lfloor np\rfloor + 1\},$$

and hence it can be shown using Stirling's formula that

$$\mu_p(\partial A) \approx \frac{1}{\sqrt{2\pi np(1-p)}}.$$

Similarly,

$$A_i = \{x \in \{0,1\}^n : \left(\sum_{i=1}^{n} x_i = \lfloor np\rfloor + 1\right) \wedge (x_i = 1)\},$$

and thus,

$$\mu_p(A_i) \approx \sqrt{\frac{p}{2\pi n(1-p)}}.$$

Therefore,

$$\text{L.h.s. of (10) } = \mu_p(\partial A)\sum_{i=1}^{n}\mu_p(A_i) \approx n \cdot \frac{1}{\sqrt{2\pi np(1-p)}} \cdot \sqrt{\frac{p}{2\pi n(1-p)}} = \Theta(1).$$

In the right hand side, we have $\sum_{i=1}^{n} \mu_p(A_i)^2 \approx \frac{p}{2\pi(1-p)}$, and thus,

$$\psi\left(3\lfloor\log(1/p)\rfloor \sum_{i=1}^{n} \mu_p(A_i)^2\right) \approx c(\log(1/p))^{\alpha}.$$

Since $\varphi\left(\mu_p(A)(1-\mu_p(A))\right) = \Theta(1)$, it follows that

$$\text{R.h.s. of (10)} \approx \frac{c}{\lfloor\log(1/p)\rfloor} \cdot (\log(1/p))^{\alpha} = c'(\log(1/p))^{\alpha-1}.$$

Therefore, if Theorem 3.6 holds for $\alpha = 1$ as conjectured by Talagrand [28], then the assertion of Proposition 3.7 is tight, up to constant factors.[8]

We note that in [28], Talagrand remarked that Theorem 3.6 can be generalized to a biased measure, but he refrained from doing so since it would have required "to write in the case $p \neq 1/2$ the proof of the lengthy Lemma 2.2 below". Our proof shows that the statement of the theorem can be generalized to the biased measure "automatically", avoiding the need to repeat the proof.

### 3.3 Upper Bound on the $d$-th Level Fourier-Walsh Coefficients

In [27], Talagrand obtained the following upper bound on the second level Fourier-Walsh coefficients of a monotone Boolean function in terms of its influences.[9]

**Theorem 3.8** (Talagrand)**.** *Consider the discrete cube $\{0,1\}^n$ endowed with the uniform measure. Let $f : \{0,1\}^n \to \{0,1\}$ be a monotone function. Then*

$$\sum_{|S|=2} \hat{f}(S)^2 \leq c \sum_{i=1}^{n} I_i(f)^2 \cdot \log\left(\frac{e}{\sum_{i=1}^{n} I_i(f)^2}\right),$$

*where $c$ is a universal constant.*

Talagrand used Theorem 3.8 as a central lemma in proving Theorem 3.6 and Theorem 3.11 (see below). In [2], Benjamini et al. generalized Talagrand's theorem to bound the $d$-th level coefficients:

**Theorem 3.9** (Benjamini, Kalai, and Schramm)**.** *Consider the discrete cube $\{0,1\}^n$ endowed with the uniform measure. Let $f : \{0,1\}^n \to \{0,1\}$ be a monotone function. Then for any $d \geq 2$,*

$$\sum_{|S|=d} \hat{f}(S)^2 \leq C_d \sum_{i=1}^{n} I_i(f)^2 \cdot \log^{d-1}\left(\frac{e}{\sum_{i=1}^{n} I_i(f)^2}\right),$$

*where $C_d$ is a constant depending only on $d$.*

Theorem 3.9 was used by Benjamini et al. to show a qualitative relation between noise sensitivity of Boolean functions and their influences, and has applications in the study of percolation.

Using Theorem 1.2, we obtain the following generalization of Theorem 3.9 to the biased measure:

[8] We note that we are not aware of a non-trivial example demonstrating the tightness of Talagrand's conjectured assertion for the uniform measure.

[9] Actually, Talagrand proved a decoupled version of the theorem, bounding $\sum_{|S|=2} \hat{f}(S)\hat{g}(S)$ for monotone functions $f, g$. Our generalization holds for the decoupled version as well.

**Proposition 3.10.** *Consider the discrete cube $\{0,1\}^n$ endowed with the measure $\mu_p$, $p \leq 1/2$. Let $f : \{0,1\}^n \to \{0,1\}$ be a monotone function. Then for any $d \geq 2$,*

$$\sum_{|S|=d} \hat{f}(S)^2 \leq C_d \cdot \left(\frac{1-p}{p\lfloor\log(1/p)\rfloor}\right)^{d-1} \cdot p(1-p) \cdot \sum_{i=1}^{n} I_i(f)^2 \cdot \log^{d-1}\left(\frac{e}{p^2\lfloor\log(1/p)\rfloor\sum_{i=1}^{n} I_i(f)^2}\right),$$

*where $C_d$ is a constant depending only on $d$.*

**Proof:** We have

$$\begin{aligned}\sum_{|S|=d} \hat{f}(S)^2 &\leq \left(\frac{1-p}{p\lfloor\log(1/p)\rfloor}\right)^{d} \sum_{|S|=d} \hat{g}(S)^2 \\ &\leq \left(\frac{1-p}{p\lfloor\log(1/p)\rfloor}\right)^{d} \cdot C_d \cdot \sum_{i=1}^{mn} I_i(g)^2 \cdot \log^{d-1}\left(\frac{e}{\sum_{i=1}^{mn} I_i(g)^2}\right) \\ &\leq \left(\frac{1-p}{p\lfloor\log(1/p)\rfloor}\right)^{d-1} \cdot C'_d \cdot p(1-p) \cdot \sum_{i=1}^{n} I_i(f)^2 \cdot \log^{d-1}\left(\frac{e}{p^2\lfloor\log(1/p)\rfloor} \cdot \sum_{i=1}^{n} I_i(f)^2\right).\end{aligned}$$

The first inequality follows from Theorem 1.2, the second follows from Theorem 3.9, and the third one follows from combination of Theorem 1.2 and Proposition 2.6. □

We note that in a recent paper [21], Theorem 3.9 was generalized to non-monotone functions, and the dependence of $C_d$ on $d$ was determined. Our generalization applies to these results as well, and the resulting upper bound for the measure $\mu_p$ is slightly weaker than the bound obtained in [21] by adaptation of the entire proof to the biased measure.

### 3.4 Lower Bound on the Correlation Between Monotone Families

In [27], Talagrand obtained a lower bound on the correlation between monotone families, improving over the classical Harris-Kleitman correlation inequality:

**Theorem 3.11** (Talagrand)**.** *Consider the discrete cube $\{0,1\}^n$ endowed with the uniform measure $\mu$. For any pair of monotone families $A, B \subset \{0,1\}^n$,*

$$\mu(A \cap B) - \mu(A)\mu(B) \geq c\varphi\left(\sum_{i=1}^{n} I_i(A)I_i(B)\right),$$

*where $\varphi(x) = x/\log(e/x)$, and $c$ is a universal constant.*

Using Theorem 1.2, we get the following generalization:

**Proposition 3.12.** *Consider the discrete cube $\{0,1\}^n$ endowed with the measure $\mu_p$, $p \leq 1/2$. For any pair of monotone families $A, B \subset \{0,1\}^n$,*

$$\mu_p(A \cap B) - \mu_p(A)\mu_p(B) \geq c\varphi\left(\lfloor\log(1/p)\rfloor \sum_{i=1}^{n} I_i(A)I_i(B)\right),$$

*where the influences are w.r.t. the measure $\mu_p$, $\varphi(x) = x/\log(e/x)$, and $c$ is a universal constant.*

Since for any monotone function $f$, $\hat{f}(\{i\}) = -\sqrt{p(1-p)}I_i(f)$, the proposition follows immediately from Theorem 1.2 using the monotonicity of the function $\varphi(x)$ in $(0,1]$.

We note that a slightly better result was obtained in [20] by transforming Talagrand's proof of Theorem 3.11 to the biased measure $\mu_p$.

## 4 Discussion

The standard reduction discussed in this paper was considered and used in numerous papers. It was first suggested in [5], as a transformation from a function $f : [0,1]^n \to \{0,1\}$ to a function $g : \{0,1\}^{mn} \to \{0,1\}$. The reduction was used there to generalize the KKL theorem [17] to the continuous cube endowed with the Lebesgue measure by obtaining a lower bound on the maximal influence. In ([12], Theorem 3.1) the reduction was adapted to the biased measure on the discrete cube (yielding Theorem 2.5 above), and used to prove that any monotone graph property has a sharp threshold. In [11] the technique of [5] was simplified and extended to the context of influences toward one and zero. In [3] the reduction was applied to a product measure on $\{0,1,2\}^n$ and used in computing the critical probability of random Voronoi percolation in the plane. In [13] the technique of [5] was generalized to arbitrary product spaces satisfying some separability condition. Finally, in [19] the reduction was used to generalize several statements from the discrete cube to the continuous cube: a lower bound on the vector of influences obtained by Talagrand [26], Friedgut's theorem characterizing functions with a low sum of influences [10], and a lower bound on the size of the boundary of monotone subsets of the discrete cube due to Talagrand [25] (which is weaker than Theorem 3.6 obtained by Talagrand [28] later).

The common feature in all these results is that they consider only *monotone two-valued* functions, and the only relation between $f$ and $g$ they obtain is a *lower bound* on the vector of influences. Our paper shows that when we restrict ourselves to the biased measure on the discrete cube (rather than the Lebesgue measure on the continuous cube and other measure spaces considered in the above papers), the reduction can be proved to be much more powerful. We obtain a direct relation between the Fourier-Walsh coefficients of $f$ and $g$, that enables us to consider *general* functions (i.e., not necessarily monotone or two-valued), and to obtain *upper bounds* in addition to the previously known lower bounds.

It is interesting to find out whether the results of our paper can be leveraged to the general reduction considering the continuous cube with the Lebesgue measure. It seems that at least some of the results do not generalize. For example, the characterization of functions with a low sum of influences, that was proved by Friedgut [10] for the biased measure on the discrete cube, fails for non-monotone functions on the continuous cube, as shown by Hatami [15].

It should be mentioned that the reduction technique we consider is naturally bounded. Since the transformation from $f$ cannot lead to all possible functions $g$, the reduction does not yield a tight result if the functions for which the respective claim is tight for the uniform measure are outside the range of the transformation. For example, consider the claim

$$\sum_{i=1}^{n} \hat{f}(\{i\})^2 \leq \mathbb{E}[f](1-\mathbb{E}[f]), \tag{13}$$

that holds for any Boolean function w.r.t. any measure $\mu_p$ (as follows immediately from the Parseval identity). The claim is tight up to a constant factor for a balanced threshold function w.r.t. any measure $\mu_p$, while the reduction yields the much weaker bound

$$\sum_{i=1}^{n} \hat{f}(\{i\})^2 \leq \frac{1-p}{p\lfloor \log(1/p) \rfloor}\mathbb{E}[f](1-\mathbb{E}[f]).$$

This happens since the inequality (13) is tight only for functions having all their non-zero Fourier-Walsh coefficients on the first level (i.e., $|S| = 1$), while the function $g$ has at most

fraction of $\frac{3p\lfloor\log(1/p)\rfloor}{1-p}$ of the total Fourier weight on the first level by its construction (see Remark 2.8).

This is the reason why Propositions 3.10 and 3.12 are slightly weaker than the results obtained in [21] and in [20] (respectively) by adapting the proof of the uniform measure case to the biased measure. In both cases, the upper bound we obtain by the reduction is weaker only for functions $f$ for which $\sum_{i=1}^{n} \hat{f}(\{i\})^2 \gg p\lfloor\log(1/p)\rfloor$.

The main open question left in understanding the reduction from $\mu_p$ to $\mu_{1/2}$ is whether one can obtain an effective *upper bound* on the $d$-th level Fourier-Walsh coefficients of $g$ in terms of the coefficients of $f$. Such bound for $d = 1$ is given in Remark 2.8, and similar (but more cumbersome) bounds can be computed for all "small" values of $d$. However, as $d$ grows, the exact formula becomes more complex and hard to work with. A "good" upper bound for large values of $d$ can lead to a simple generalization to the biased measure of the lower bounds on the Fourier tail of Boolean and general bounded functions [6, 9], [10] and maybe also of other results.

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